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On the Conformal Gaussian Curvature Equation in R^2

Kuo-Shung Cheng and Chang-Shou Lin

Department of Mathematics, National Chung Cheng University, Minghsiung, Chia-Yi 621, Taiwan
E-mail: kscheng@math.ccu.edu.tw
E-mail: cslin@math.ccu.edu.tw

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In this paper, we consider the equation

$$\Delta u + K(x)\, e^{2u} = 0 \quad \text{in } R^2 \tag{0.1}$$

geometric viewpoint, it is quite interesting to ask what the best possible range is of total curvature of all solutions of (0, 1). In this paper, we study this problem for radial solutions. We also construct some particular K to demonstrate the rich phenomenon. In particular, we show by examples that when $K(x)$ is negative for $|x|$ large, Eq. (0.1) possess a branch of solutions which satisfies some monotonicity property. © 1998 Academic Press

1. INTRODUCTION

In this paper, we are going to study the solution set of the equation

$$\Delta u + K(x)\, e^{2u} = 0 \qquad \text{in } \mathbf{R}^2, \tag{1.1}$$

where $\Delta = \sum_{i=1}^{2} (\partial^2/\partial x_i^2)$ and K is a locally bounded function in $\mathbf{R}^2$. To say that u is a solution of (1.1), we mean that u is in $W^{2,s}_{loc}(\mathbf{R}^2)$ for any $s > 1$ and satisfies Eq. (1.1) in the distributional sense. One of the motivations in studying Eq. (1.1) arises from the problem of prescribing Gaussian curvature in $\mathbf{R}^2$. Let K be a given function in $\mathbf{R}^2$, one would like to find a metric g conformal to the flat metric $|dx|^2$ such that $K/2$ is the Gaussian curvature of the new metric g. Let $g = e^{2u}\,|dx|^2$ for some function u in $\mathbf{R}^2$. Then the question above is equivalent to finding a solution of Eq. (1.1). For the history and background material, we refer the reader to [CN1] and references therein.

The first nonexistence result concerning Eq. (1.1) is found in Ahlfors [A] in 1938. He proved that in the case of $K(x) \equiv -1$ in $\mathbf{R}^2$, Eq. (1.1) does not

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have any solution in the entire space $\mathbf{R}^2$. His result was later improved by Sattinger [S], Oleinik [O], Cheng and Lin [CL], and many others. In particular, we want to mention the result of Cheng–Lin [CLn3]. To state the result, we first introduce a quantity $\alpha_1 = \alpha_1(K)$

$$\alpha_1 \equiv \sup \left\{ \alpha \,\middle|\, \int_{\mathbf{R}^2} |K(x)| \, (1+|x|^2)^\alpha \, dx < +\infty \right\}. \tag{1.2}$$

THEOREM A. *Suppose that $K(x)$ is nonpositive in $\mathbf{R}^2$ and satisfies $\alpha_1(K) > 0$. Then Eq.* (1.1) *possesses infinitely many solutions. If, in addition, K satisfies*

$$|x|^{-m} \leqslant |K(x)| \leqslant |x|^m \tag{1.3}$$

for $|x|$ large, where m is a positive constant, then the following statements hold.

(i) *If $\alpha_1(K) \leqslant 0$, then Eq.* (1.1) *does not possess any solution in $\mathbf{R}^2$.*

(ii) *For $0 < \alpha < \alpha_1$, Eq.* (1.1) *possesses a unique solution $u_\alpha(x)$ such that*

$$u_\alpha(x) = \alpha \log |x| + O(1) \qquad \text{at } \infty. \tag{1.4}$$

(iii) *Set $U(x) = \sup\{u(x) \mid u \text{ is a solution of } (1.1)\}$. Then $U(x)$ is well-defined in $\mathbf{R}^2$ and is a solution of* (1.1). *Furthermore for any $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that for $|x| \geqslant R$,*

$$(\alpha_1 - \varepsilon) \log |x| \leqslant U(x) \leqslant \alpha_1 \log |x| + C, \tag{1.5}$$

where C is a constant independent of ε.

(iv) *For any solution u of* (1.1), *one has either $u \equiv u_\alpha$ for some $0 < \alpha < \alpha_1$ or $u \equiv U$.*

(v) *For $0 < \alpha < \beta < \alpha_1$, $u_\alpha(x) < u_\beta(x) < U(x)$ in $\mathbf{R}^2$.*

We note that the existence of the maximal solution $U(x)$ was proved in [CN1] and part of Theorem A was obtained in [CN1, CN2] under some stronger conditions on K. Theorem A is a special case of Theorems 1.1 and 1.4 in [CLn3]. From Theorem A, the solution set to Eq. (1.1) is understood quite well for the case $K(x) \leqslant 0$ in $\mathbf{R}^2$.

There is a geometric interpretation for solutions satisfying (1.5). Following conventional notations, *a solution u is said to have a finite total curvature* if $Ke^{2u} \in L^1(\mathbf{R}^2)$. In this case, the quantity $\int Ke^{2u}$ is defined as the total curvature for the solution u. It was proved in [CLn1] that if (1.3) holds for $|x|$ large, than u is a solution with a finite total curvature if and only if the limit $u(x)/\log|x|$ exists as $|x| \to +\infty$. In this case,

$$\lim_{|x| \to +\infty} \left(\frac{u(x)}{\log |x|} \right) = \frac{-1}{2\pi} \int_{\mathbf{R}^2} K(x)\, e^{2u(x)}\, dx \tag{1.6}$$

holds always.

For the case that $K(x)$ is positive somewhere in $\mathbf{R}^2$, McOwen [M] proved

THEOREM B. *Suppose $K(x)$ is positive somewhere in $\mathbf{R}^2$ and $K(x) = O(|x|^{-l})$ for some $l > 0$. Then for any $-2 < \alpha < \min(0, l-2)$, Eq. (1.1) possesses a solution $u(x)$ satisfying*

$$u(x) = \alpha \log |x| + O(1) \qquad \text{at } \infty.$$

Clearly from (1.6), solution $u(x)$ obtained in Theorem B has a positive total curvature. If $K(x) \leqslant 0$, for $|x|$ large, Theorem B was extended by Cheng and Lin [CLn2]. They proved that if $-|x|^m \leqslant K(x) \leqslant 0$ for $|x|$ large and $\alpha_1(K) > -2$, then for any $-2 < \alpha < \min(0, \alpha_1(K))$, Eq. (1.1) possesses a solution $u(x)$ such that

$$u(x) = \alpha \log |x| + O(1) \qquad \text{at } \infty.$$

Note that $|K(x)|$ could be allowed to behave like $|x|^l$ for some $0 \leqslant l < 2$. As far as the authors know, this is the only result where $|K(x)|$ may be allowed to grow at infinity. Meanwhile, in [CLn4], the authors show by example that for the case $K(x) \geqslant 0$ in $\mathbf{R}^2$, the range of α in Theorem B is the best possible. From the viewpoint of geometry, it is quite interesting to ask for the possible range of α, i.e., the possible range of total curvatures of solutions of Eq. (1.1). The purpose of this paper is devoted to studying this problem, although we do consider the case when K and solutions are radially symmetric functions only. Our first result is the following theorem, which for simplicity, $K(r)$ is assumed to be non-negative.

Following conventional notations, a solution $u(r; a)$ denotes the unique solution of the following initial value problem:

$$\begin{cases} u'' + \dfrac{1}{r} u' + K(r)\, e^{2u(r)} = 0 \\ u(0; a) = a \qquad \text{and} \qquad u'(0; a) = 0. \end{cases}$$

Since $K(r) \geqslant 0$ in $\mathbf{R}^2$, it is not difficult to see that for any $a \in \mathbf{R}$, $u(r; a)$ exists for all $r > 0$ and always has a finite total curvature. Thus,

$$u(r; a) = \alpha(a) \log r + b(a) + o(1) \tag{1.8}$$

for r large, where $\alpha(a) < 0$, $b(a) \in \mathbf{R}$, and $o(1) \to 0$ as $r \to +\infty$. Since $Ke^{2u} \in L^1(\mathbf{R}^2)$ for each a, it can be proved without too much difficulty that for any compact set of a, there exists a $\varepsilon_0 > 0$ such that

$$\alpha(a) + q < -(1 + \varepsilon_0),$$

provided that $K(r) = r^{2q}(K(\infty) + o(1))$ for r large and for some $K(\infty) > 0$. Thus from (1.8), we conclude that both $\alpha(a)$ and $b(a)$ are continuous in a.

THEOREM 1.1. *Suppose that $K(r) = K(0)\, r^{2p} + O(r^{2p+k})$ near $r = 0$ and $K(r) = K(\infty)\, r^{2q} + O(r^{2q-l})$ near $r = \infty$ for some positive constants $K(0)$, $K(\infty)$, k, l, and for some real numbers $p > -1$ and q. Then the following statements hold.*

(i) *If $q \leqslant -1$, then*

$$\lim_{a \to -\infty} \alpha(a) = 0, \quad \lim_{a \to +\infty} \alpha(a) = -2(1+p), \quad \textit{and} \quad \lim_{a \to \pm\infty} b(a) = -\infty.$$

(ii) *If $-1 < q < 1 + 2p$, then*

$$\lim_{a \to -\infty} \alpha(a) = -2(1+q), \quad \lim_{a \to +\infty} \alpha(a) = -2(1+p), \quad \textit{and}$$

$$\lim_{a \to \pm\infty} b(a) = \mp\infty.$$

(iii) *If $1 + 2p \leqslant q$, then*

$$\lim_{a \to -\infty} \alpha(a) = -2(1+q), \quad \lim_{a \to +\infty} \alpha(a) = 2(p-q), \quad \textit{and}$$

$$\lim_{a \to \pm\infty} b(a) = +\infty.$$

Let $\alpha_\pm = \lim_{a \to \pm\infty} \alpha(a)$ in Theorem 1.1, and let I denote the open interval bounded by α_+ and α_-. Naturally, one might ask whether the interval I is the best possible range of α. In the following, we will show by some particular function K that I is the set consisting of these α. For this purpose, we let

$$K(r) = \begin{cases} r^{2p} & \text{if} \quad r \leqslant 1 \\ r^{2q} & \text{if} \quad r \geqslant 1, \end{cases} \tag{1.9}$$

where $p > -1$ and $q \in \mathbf{R}$.

THEOREM 1.2. *Let K be defined in* (1.9) *with $p \neq q$. Then for any $\alpha \in I$, there exists a unique radial solution u_α of* (1.1) *such that*

$$u_\alpha(r) = \alpha \log r + O(1) \qquad \text{at } \infty.$$

Furthermore, if u is any radial solution of (1.12), *then $u(r) \equiv u_\alpha(r)$ for some $\alpha \in I$.*

Remark 1.3. If $-1 < p \leqslant 0$ and $q \leqslant 0$, then $K(r)$ is decreasing in r. In [CLn1], the authors proved that if u is a solution with a finite total curvature, then $u(x)$ is radially symmetric. Thus, the solution u_α in Theorem 1.2 classifies all possible solutions with a finite total curvature in this case. In particular, for $p = 0$ and $q < 0$, $I = (-2, \min(0, -(2+2q))$. Thus, this demonstrates that the result obtained by McOwen is the best possible.

It is quite surprising that the uniqueness holds for the case of positive total curvature, at least for the functions K in Theorems 1.2 and 1.4 below, where K changes sign in $\mathbf{R}^2$. At a moment of reflection, we can show that when $K(r)$ is positive somewhere and Eq. (1.1) possesses a solution with a negative total curvature, then (1.1) possesses at least two solutions with the same value of negative total curvature. This is a new phenomenon, which we should come back to study in a future work. To illustrate this, we consider a particular K, which is defined in the following.

Let

$$K(r) = \begin{cases} K_0 & \text{if } \quad r \leqslant 1, \\ -r^{-2m} & \text{if } \quad r > 1, \end{cases} \tag{1.10}$$

where K_0 is a positive constant and $m \in \mathbf{R}$.

THEOREM 1.4. *Let K be defined in* (1.10). *Then*

(i) *If $m \leqslant -1$, Eq.* (1.1) *possesses no radial solutions, i.e., for any $a \in \mathbf{R}$, $u(r; a)$ blows up at a finite r.*

(ii) *If $-1 < m \leqslant 1$, then there exists an $a_0 \in \mathbf{R}$ such that $\alpha(a)$ is monotonically decreasing in $a \in [a_0, \infty)$ with $\lim_{a \to +\infty} \alpha(a) = -2$ and $\lim_{a \to a_0} \alpha(a) = m - 1$. Furthermore, $u(r; a)$ blows up at a finite r for each $a < a_0$. Thus, there exists no entire solution of* (1.1) *with negative total curvature.*

(iii) *If $1 < m < \sqrt{(1+K_0)/K_0}$, then there exist $a_0 < a_1$ such that $\alpha(a)$ is monotonically increasing in $(-\infty, a_0]$ and is monotonically decreasing in $[a_1, +\infty)$ with*

$$\lim_{a \to -\infty} \alpha(a) = 0 \qquad \textit{and} \qquad \lim_{a \to a_0^-} \alpha(a) = m - 1,$$

$$\lim_{a \to +\infty} \alpha(a) = -2 \qquad \textit{and} \qquad \lim_{a \to a_1^+} \alpha(a) = m - 1.$$

For $\alpha \in (a_0, a_1)$, $u(r; a)$ *blows up at a finite* r. *Furthermore, if* $\alpha(a) = \alpha(a^*)$ *for* $a < a_0 < a_1 < a^*$, *we have*

$$\begin{cases} u(r; a) < u(r; a^*), & \textit{and} \\ u(r; a) < u(r; a'), & \textit{for} \quad a < a' \leqslant a_0. \end{cases}$$

(iv) *If* $m = \sqrt{(1 + K_0)/K_0}$, *then results similar to* (iii) *hold except that* $a_0 = a_1$.

(v) *If* $\sqrt{(1 + K_0)/K_0} < m < (1 + K_0)/K_0$, *then results similar to* (iii) *hold except that* $a_0 = a_1$ *and*

$$0 < \lim_{a \to a_0} \alpha(a) < m - 1.$$

(vi) *If* $m > (1 + K_0)/K_0$, *then* $\alpha(a)$ *is monotonically decreasing for all* $a \in \mathbf{R}$ *such that*

$$\lim_{a \to -\infty} \alpha(a) = 0 \qquad \textit{and} \qquad \lim_{a \to +\infty} \alpha(a) = -2.$$

Remark 1.5. First we note that $\alpha_1(K) = m - 1$. For $m < 1$, solutions with $\alpha \in (-2, \alpha_1)$ where obtained in [CLn2], as mentioned before. We see from the statements of Theorem 1.4 that uniqueness holds for solutions with positive total curvature, but uniqueness fails for the case of negative total curvature. Another interesting result in Theorem 1.4 is that there exists a branch of solutions $u(r; a)$ with negative total curvature such that the monotone property of (v) of Theorem 1.1 still holds. This is unexpected because the maximum principle can not be applied in general when K is positive somewhere.

In Theorem 1.1, the case $p = q = 0$ is particularly interesting. In this case,

$$\lim_{a \to -\infty} \alpha(a) = -2 = \lim_{a \to \infty} \alpha(a).$$

Thus in the following we consider the case when K is bounded by two positive constants. We define

$$G(r) = r^2 K(r) - 2 \int_0^r sK(s)\, ds. \tag{1.11}$$

Our results are

THEOREM 1.6. *Suppose that* $G(r) \geqslant 0$ *for* $r > 0$ *and* $G \not\equiv 0$. *Then* $\alpha(a) < -2$ *for all* $a \in \mathbf{R}$.

THEOREM 1.7. *Suppose that* $G(r) \leqslant 0$ *for* $r > 0$ *and* $G(r) \not\equiv 0$. *Then* $-2 < \alpha(a) < -1$ *for all* $a \in \mathbf{R}$.

THEOREM 1.8. *Suppose that* $K(r) = K(0) + K_1(0)\, r^k + o(r^k)$ *for* r *small for some* $0 < k \leqslant 2$, $K(0) > 0$, $K_1(0) > 0$ *and there exist* $R > 0$ *such that* $G(R) < 0$, $G(r) \leqslant G(R)$ *for* $r \geqslant R$. *Then there exists an* a_0 *such that* $\alpha(a_0) = -2$.

THEOREM 1.9. *Suppose that* $K(r) = K(0) - K_1(0)\, r^k + o(r^k)$ *for* r *small for some* $0 < k < 2$, $K(0) > 0$, $K_1(0) > 0$ *and there exist* $R > 0$ *such that* $G(R) > 0$, $G(r) \geqslant G(R)$ *for* $r \geqslant R$. *Then there exists an* a_0 *such that* $\alpha(a_0) = -2$.

If K is C^1, then from (1.11) we see that

$$G(r) = \int_0^r s^2 K'(s)\, ds. \tag{1.12}$$

Thus if K is nondecreasing on $(0, \infty)$, then $G(r) \geqslant 0$ for all $r > 0$ and if K is nonincreasing on $(0, \infty)$, then $G(r) \leqslant 0$ for all $r > 0$. Thus from Theorem 1.6, we know that if K is nondecreasing on $(0, \infty)$ and is not identically a constant, then $\alpha(a) < -2$ and $\alpha(a) \to -2$ as $a \to \pm\infty$. It seems not unreasonable to conjecture that $\alpha(a)$ has a simple structure such as $\alpha(a)$ is monotonically decreasing on $(-\infty, a_0)$ and is monotonically increasing on (a_0, ∞) for some a_0. Similarly, if K is nonincreasing on $(0, \infty)$ and is not identically a constant, then $-2 < \alpha(a) < -1$ and $\alpha(a) \to -2$ as $a \to \pm\infty$. We also expect that $\alpha(a)$ has a simple structure. The answer to these questions is quite a surprise. We have

THEOREM 1.10. *For a given positive integer* n, *we can find a nondecreasing function* $K_1(r)$ *and a nonincreasing function* $K_2(r)$ *such that the corresponding functions* $\alpha_1(a)$ *and* $\alpha_2(a)$ *have at least* n *local minima and* n *local maxima, respectively.*

Thus, as we can see, even if K is radial and monotonic, the structure of the solution set is still very complicated.

This paper is organized as follows. In Section 2, we prove Theorem 1.1 and Theorems 1.6–1.9. In Section 3, a complete set of radial solutions for Theorems 1.2 and 1.4 is constructed. Theorem 1.10 is proved in Section 4.

2. PROOFS OF THEOREMS 1.1, 1.6–1.9

In this section, we will give proofs for Theorems 1.1, 1.6–1.9. First we want to prove Theorem 1.1. We need to consider the following problem

$$\begin{cases} u'' + \dfrac{1}{r} u' + K(r)\, e^{2u} = 0, & r \in (0, 1] \\ u(r) = A \log r + a + o(1) & \text{as} \quad r \to 0, \end{cases} \tag{2.1}$$

where A and a are constants.

Our first lemma is

LEMMA 2.1. *Suppose that $K(r) = K(0)\, r^{2p} + O(r^{2p+k})$ for r near 0, where $K(0) > 0$, $k > 0$ and p are constants. If $A \leqslant -(1+p)$, then (2.1) admits no solution.*

Proof. Let $u(r) = A \log r + a + v(r)$. Then v satisfies

$$\begin{cases} u'' + \dfrac{1}{r} v' + K(r)\, r^{2A} e^{2a} e^{2v} = 0, & r \in (0, 1] \\ v(r) = o(1) & \text{as} \quad r \to 0. \end{cases} \tag{2.2}$$

If $A \leqslant -(1+p)$, then

$$K(r)\, r^{2A} = K(0)\, r^{2(p+A)} + O(r^{2(p+A)+k})$$

for r near 0, where $2(p+A) \leqslant -2$. Thus, we conclude that (2.2) admits no solution. This proves the Lemma.

Our second lemma is

LEMMA 2.2. *Suppose that K satisfies the assumption in Lemma 2.1 and $-(1+p) < A$. Then, for each such A and each real constant a, (2.1) possesses a unique solution $u = u(r, a, A)$. Furthermore, $u(r, a, A)$ is continuous as a function of a, and A and satisfies*

$$\begin{cases} u(1, a, A) = a + O(e^{2a}), \\ u'(1, a, A) = A + O(e^{2a}) \end{cases} \tag{2.3}$$

for $a \leqslant -M$ and

$$\begin{cases} u(1, a, A) = -a + C + O(e^{-\mu a}), \\ u'(1, a, A) = -(A + 2 + 2p) + O(e^{-\mu 2a}) \end{cases} \tag{2.4}$$

for $M \leqslant a$, where M is a large constant, C is a constant independent of a, and μ is a constant satisfying $0 < \mu \leqslant 2$.

Proof. We let

$$u(r) = A \log r + v(r). \tag{2.5}$$

Then $v(r)$ satisfies the initial value problem

$$\begin{cases} v'' + \dfrac{1}{r} v' + K(r)\, r^{2A} e^{2v} = 0, & r \in (0, 1] \\ v(0) = a. \end{cases} \tag{2.6}$$

The proof of the existence and the uniqueness of the solution to (2.6) is elementary. We should omit the proof. Integrating (2.6), one has

$$v(r) = a - \int_0^r s \log\left(\frac{r}{s}\right) K(s)\, s^{2A} e^{2v(s)}\, ds, \qquad r \in [0, 1]. \tag{2.7}$$

Since the function

$$\int_0^r s \log\left(\frac{r}{s}\right) K(s)\, s^{2A}\, ds$$

is bounded in $[0, 1]$, we conclude that

$$\begin{cases} v(1) = a + O({}^{2a}), \\ v'(1) = O(e^{2a}) \end{cases} \tag{2.8}$$

for $a \leqslant -M$ if M is sufficiently large. This proves (2.3).

To prove (2.4), we let

$$\tilde{u}(r) = (A_1 - 1 - p) \log r + a - \log[1 + B_1 r^{2A_1}], \tag{2.9}$$

and

$$u(r) = \tilde{u} + v(r). \tag{2.10}$$

Then $\tilde{u}(r)$ and $v(r)$ satisfy

$$\tilde{u}'' + \frac{1}{r} \tilde{u}' + K(0)\, r^{2p} e^{2\tilde{u}} = 0, \tag{2.11}$$

$$\begin{cases} v'' + \dfrac{1}{r} v' + K(r)\, e^{2\tilde{u}} e^{2v} - K(0)\, r^{2p} e^{2\tilde{u}} = 0, \\ v(0) = 0, \end{cases} \tag{2.12}$$

where $A_1 = A + (1+p) > 0$ and $B_1 = K(0)\, e^{2a}/4A_1^2 > 0$. Thus v satisfies the following integral equation

$$v(r) = g(r) - \int_0^r s \log\left(\frac{r}{s}\right) K(s)\, e^{2\tilde{u}(s)}(e^{2v(s)} - 1)\, ds, \tag{2.13}$$

where

$$g(r) = -\int_0^r s \log\left(\frac{r}{s}\right) [K(s) - K(0)\, s^{2p}]\, e^{2\tilde{u}(s)}\, ds. \tag{2.14}$$

Now we choose δ small such that

$$|K(r) - K(0)\, r^{2p}| \leqslant Cr^{2p+k} \tag{2.15}$$

for $0 \leqslant r \leqslant \delta$. Let

$$B_1 r^{2A_1} = t^{2A_1}. \tag{2.16}$$

Then $r = \gamma t$ with $\gamma = (2A_1 e^{-a}/\sqrt{K(0)})^{1/A_1}$.

Let $v(r) = y(t)$. Then y satisfies

$$y(t) = g(\gamma t) - \int_0^t s \log\left(\frac{t}{s}\right) 4A_1^2[1 + O(\gamma s)^k]\, \frac{s^{2A_1-2}(e^{2y} - 1)}{(1 + s^{2A_1})^2}\, ds. \tag{2.17}$$

Consider $0 \leqslant t \leqslant \gamma^{-1}\delta$. We have, for a large,

$$\begin{aligned}
|g(\gamma t)| &\leqslant \int_0^t s \log\left(\frac{t}{s}\right) \cdot C \cdot e^{-ka/A_1} \cdot \frac{s^{2A_1-2+k}}{(1+s^{2A_1})^2}\, ds \\
&\leqslant C \cdot \begin{cases} e^{-ka/A_1} \cdot a & \text{if } \ 0 < k < 2A_1, \\ e^{-ka/A_1} \cdot a^2 & \text{if } \ k = 2A_1, \\ e^{-2a} \cdot a & \text{if } \ k > 2A_1, \end{cases} \\
&\leqslant C \cdot e^{-\mu a} \qquad \text{for some } \quad \mu \leqslant 2.
\end{aligned} \tag{2.18}$$

Let $|y(t)| = z(t)$. The from (2.17) and (2.18) we obtain

$$\begin{aligned}
z(t) &\leqslant Ce^{-\mu a} + 4 \int_0^t s \log\left(\frac{t}{s}\right) A_1^2[1 + O(\gamma s)^k]\, \frac{s^{2A_1-2}}{(1+s^{2A_1})^2} \\
&\quad \left|\frac{e^{2y(s)} - 1}{y(s)}\right| z(s)\, ds \leqslant Ce^{-\mu a} + 4A_1^2[1+\epsilon] \cdot 2[1+\epsilon'] \int_0^t s \log\left(\frac{t}{s}\right) \\
&\quad \times \frac{s^{2A_1-2}}{(1+s^{2A_1})^2}\, z(s)\, ds,
\end{aligned} \tag{2.19}$$

where ϵ can be made small if we take δ small and ϵ' can be made small if we can prove that $z(t)$ is small when a is large. The arguments are now very similar to the proofs in [CS, Lemma 2.9 ad Theorem 2.6]. We consider $\tilde{w}$ and w satisfying

$$\begin{cases} \tilde{w}(t) \leqslant \tilde{a} + \tilde{b} \int_0^t s \log\left(\frac{t}{s}\right) \frac{s^{2A_1-2}\tilde{w}(s)}{(1+s^{2A_1})^2}\, ds, \\ w(t) = \tilde{a} + \tilde{b} \int_0^t s \log\left(\frac{t}{s}\right) \frac{s^{2A_1-2}w(s)}{(1+s^{2A_1})^2}\, ds, \end{cases} \tag{2.20}$$

where $\tilde{a}$ and $\tilde{b}$ are two constants satisfying $\tilde{a} > 0$ and $0 < \tilde{b} < 16A_1^2$. We claim that

$$\begin{cases} \tilde{w}(t) \leqslant w(t) \\ w(t) \leqslant \tilde{a}(C_1 + C_2 \log(1+t)) \end{cases} \tag{2.21}$$

for $t \geqslant 0$ where C_1 and C_2 are two constants depending on $\tilde{b}$. We only sketch the proof. We let $X = \{\text{continuous functions on } [0, \infty)\}$ and

$$Y = \{ w \in X \mid \tilde{a} \leqslant w(t) \leqslant g(t),\ t \geqslant 0 \},$$

where $g(t) = \tilde{a}(1 + Bt^{2A_1-\beta})$. Define

$$(Tw)(t) = \tilde{a} + \tilde{b} \int_0^t s \log\left(\frac{t}{s}\right) \frac{s^{2A_1-2}w(s)}{(1+s^{2A_1})^2}\, ds. \tag{2.22}$$

Choose $\eta > 0$ and $\beta > 0$ so small to make $\tilde{b} < 4(1-\eta)(2A_1-\beta)^2$ and choose B so large to make

$$\eta B (2A_1 - \beta)^2\, (1 + t^{2A_1})^2 \geqslant \tilde{b} t^{\beta}$$

for all $t \geqslant 0$. Then, we let $h(t) = g(t) - (Tg)(t)$ and conclude that

$$\begin{cases} h'(t) + \dfrac{1}{t} h'(t) \geqslant 0, \\ h(0) = 0,\ h'(0) = 0. \end{cases}$$

Hence $h(t) \geqslant 0$ and $Tg \leqslant g$. This proves that $TY \subset Y$. Hence T has a fixed point w_0 in Y, i.e.,

$$w_0(t) = \tilde{a} + \tilde{b} \int_0^t s \log\left(\frac{t}{s}\right) \frac{s^{2A_1-2}w_0(s)}{(1+s^{2A_1})^2}\, ds.$$

Now the proof of (2.21) follows almost the same proof as in [CS, Lemma 2.9]. We omit it.

Thus we conclude from (2.19), (2.20), and (2.21) that

$$|v(r)| \leqslant Ce^{-\mu a}, \qquad r \in [0, \delta] \tag{2.23}$$

for some μ, $0 < \mu \leqslant 2$ and $M \leqslant a$, M large. Using (2.9) and (2.10), we obtain

$$\begin{cases} u(\delta) = -a + C + O(e^{-\mu a}), \\ u'(\delta) = -\dfrac{1}{\delta}(A_1 + 1 + p) + O(e^{-\mu a}) \end{cases} \tag{2.24}$$

for $M \leqslant a$. Now for $\delta \leqslant r \leqslant 1$, we have

$$\begin{aligned} u'(1) &= \delta u'(\delta) - \int_\delta^1 sK(s)\, e^{2u(s)}\, ds \\ &= -(A_1 + 1 + p) + O(e^{-\mu a}), \\ u(1) &= u(\delta) + \delta u'(\delta) \log \frac{1}{\delta} - \int_\delta^1 s \log\left(\frac{1}{s}\right) K(s)\, e^{2u(s)}\, ds \\ &= -a + C + O(e^{-\mu a}), \end{aligned}$$

for $M \leqslant a$. This proves (2.4) and the lemma. Q.E.D.

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We decompose Eq. (1.7) into

$$\begin{cases} u'' + \dfrac{1}{r} u' + K(r)\, e^{2u} = 0, \qquad \text{for} \quad r \in (0, 1], \\ u(0) = a, \end{cases} \tag{2.25}$$

and

$$u'' + \frac{1}{r} u' + K(r)\, e^{2u} = 0, \qquad \text{for} \quad r \in [1, \infty). \tag{2.26}$$

Now from Lemma 2.2, by taking $A = 0$, we conclude that there exists a sufficiently large M such that

$$\begin{cases} u(1, a) = a + O(e^{2a}) \\ u'(1, a) = O(e^{2a}) \end{cases} \tag{2.27}$$

for $a \leqslant -M$ and

$$\begin{cases} u(1, a) = -a + C + O(e^{-\mu a}) \\ u'(1, a) = -(2+2p) + O(e^{-\mu a}) \end{cases} \tag{2.28}$$

for $a \geqslant M$.

We let $r = (1/t)$ and $u(r) = v(t) + 2 \log t$. Then (2.26) becomes

$$v'' + \frac{1}{t} v' + K\left(\frac{1}{t}\right) e^{2v} = 0, \qquad \text{for} \quad t \in (0, 1], \tag{2.29}$$

where $K(1/t) = K(\infty)\, t^{-2q} + O(t^{-2q+l})$ for t small.

We let

$$v(t) = \tilde{B} \log t + b + o(1) \qquad \text{as} \quad t \to 0. \tag{2.30}$$

Again, by using Lemma 2.2, we conclude from (2.29) and (2.30) that there exists a sufficiently large N such that

$$\begin{cases} v(1, b, \tilde{B}) = b + O(e^{2b}) \\ v'(1, b, \tilde{B}) = \tilde{B} + O(e^{2b}) \end{cases} \tag{2.31}$$

for $b \leqslant -N$ and

$$\begin{cases} v(1, b, \tilde{B}) = -b + C + O(e^{-\nu b}) \\ v'(1, b, \tilde{B}) = -(\tilde{B} + 2 - 2q) + O(e^{-\nu b}) \end{cases} \tag{2.32}$$

for $b \geqslant N$, where $0 < \nu \leqslant 2$.

However, as we know from the relation of u and v, we have $u(1) = v(1)$ and $u'(1) = -v'(1) - 2$. Thus, from (2.31) and (2.32), we obtain

$$\begin{cases} u(1, b, \tilde{B}) = b + O(e^{2b}) \\ u'(1, b, \tilde{B}) = -\tilde{B} - 2 + O(e^{2b}) \end{cases} \tag{2.33}$$

for $b \leqslant -N$ and

$$\begin{cases} u(1, b, \tilde{B}) = -b + C + O(e^{-\nu b}) \\ u'(1, b, \tilde{B}) = \tilde{B} - 2q + O(e^{-\nu b}) \end{cases} \tag{2.34}$$

for $b \geqslant N$. Now we let $\alpha = -\tilde{B} - 2$ and we conclude

$$\begin{cases} u(1, b, \tilde{B}) = b + O(e^{2b}) \\ u'(1, b, \tilde{B}) = \alpha + O(e^{2b}) \end{cases} \tag{2.35}$$

for $b \leqslant -N$ and

$$\begin{cases} u(1, b, \tilde{B}) = -b + C + O(e^{-\nu b}) \\ u'(1, b, \tilde{B}) = -[\alpha + 2 + q] + O(e^{-\nu b}) \end{cases} \tag{2.36}$$

for $b \geqslant N$.

From (2.25) we know that $(u(1, a), u'(1, a))$ is a continuous curve on the plane satisfying (2.27) and (2.28). From (2.26), (2.29), and (2.30) we know that $(u(1, b, \tilde{B}), u'(1, b, \tilde{B}))$ is also a continuous curve for each α. We let

$$\Gamma_1 = \{(u(1, a), u'(1, a)) \mid -\infty < a < \infty\}$$

and

$$\Gamma_{2, \alpha} = \{(u(1, b, \tilde{B}), u'(1, b, \tilde{B})) \mid -\infty < b < \infty\}.$$

If for some α, $\Gamma_1 \cap \Gamma_{2, \alpha} \neq \phi$. Let a_0, b_0 be such that

$$\begin{cases} u(1, a_0) = u(1, b_0, \tilde{B}) \\ u'(1, a_0) = u'(1, b_0, \tilde{B}). \end{cases}$$

Then Eq. (1.7) with $u(0) = a_0$ has a unique solution $u(r, a_0)$ satisfying

$$u(r, a_0) = \alpha \log r + b_0 + o(1) \qquad \text{as} \quad r \to \infty. \tag{2.37}$$

From our arguments, we conclude that α and b_0 are continuous functions of a. From Lemma 2.1, we see that $A > -(1+p)$ and $\alpha < -(1+q)$. Hence $\alpha < -[\alpha + 2(1+q)]$. Now we divide the proof into several cases.

Case 1. $q \leqslant -1$.

In this case $-(1+q) \geqslant 0$. Note that

$$-(1+q) = \tfrac{1}{2}\{-[\alpha + 2(1+q)] + \alpha\}.$$

Thus in this case, it is easy to see that for each α satisfying $-2(1+p) < \alpha < 0$, $\Gamma_1 \cap \Gamma_{2, \alpha} \neq \phi$. It is also easy to see from (2.27), (2.28), and (2.35) that

$$\lim_{a \to -\infty} \alpha(a) = 0, \qquad \lim_{a \to \infty} \alpha(a) = -2(1+p), \tag{2.38}$$

and

$$\lim_{a \to -\infty} b(a) = -\infty = \lim_{a \to \infty} b(a). \tag{2.39}$$

Case 2. $-1<q<p$.

In this case $-(1+p)<-(1+q)<0$. Hence for each α satisfying $-2-2p<\alpha<-2-2q$, $\Gamma_1 \cap \Gamma_{2,\alpha} \neq \phi$. It is also easy to see from (2.27), (2.28), (2.35), and (2.36) that

$$\lim_{a \to -\infty} \alpha(a) = -2-2q, \qquad \lim_{a \to \infty} \alpha(a) = -2-2p, \tag{2.40}$$

and

$$\lim_{a \to -\infty} b(a) = \infty, \qquad \lim_{a \to \infty} b(a) = -\infty. \tag{2.41}$$

Case 3. $q=p$.

In this case, there is no obvious range of α to assure $\Gamma_1 \cap \Gamma_{2,\alpha} \neq \phi$. But it is still easy to see that

$$\lim_{a \to -\infty} \alpha(a) = -2-2q = -2-2p = \lim_{a \to \infty} \alpha(a) \tag{2.42}$$

and

$$\lim_{a \to -\infty} b(a) = \infty, \qquad \lim_{a \to \infty} b(a) = -\infty. \tag{2.43}$$

Case 4. $p<q<1+2p$.

In this case, $-2(1+p)<-(1+q)<-(1+p)$. Hence for each α satisfying $-2-2q<\alpha<-2-2p$, $\Gamma_1 \cap \Gamma_{2,\alpha} \neq \phi$. It is also easy to see that

$$\lim_{a \to -\infty} \alpha(a) = -2-2q, \qquad \lim_{a \to \infty} \alpha(a) = -2-2p, \tag{2.44}$$

and

$$\lim_{a \to -\infty} b(a) = \infty, \qquad \lim_{a \to \infty} b(a) = -\infty. \tag{2.45}$$

Case 5. $1+2p \leqslant q$.

In this case, $-(1+q) \leqslant -2(1+p)$. Hence for each α satisfying $-2(1+q)<\alpha<2(p-q)$, $\Gamma_1 \cap \Gamma_{2,\alpha} \neq \phi$. It is easy to see that

$$\lim_{a \to -\infty} \alpha(a) = -2-2q, \qquad \lim_{a \to \infty} \alpha(a) = 2(p-q), \tag{2.46}$$

and

$$\lim_{a \to -\infty} b(a) = \infty, \qquad \lim_{a \to \infty} b(a) = \infty. \tag{2.47}$$

Combining (2.38) to (2.47), we prove Theorem 1.1. Q.E.D.

Now we can prove Theorems 1.6–1.9.

Proof of Theorem 1.6. We define a quantity

$$P(r; u) = ru'(ru' + 2) + r^2 K(r)\, e^{2u}, \tag{2.48}$$

where we let $u(r) = u(r, a)$. It is also easy to see that u satisfies

$$u(r) = a - \int_0^r s \log\left(\frac{r}{s}\right) K(s)\, e^{2u(s)}\, ds. \tag{2.49}$$

Hence we conclude from (2.49) that

$$\alpha(a) = -\int_0^\infty sK(s)\, e^{2u(s)}\, ds = \lim_{r \to \infty} ru'(r). \tag{2.50}$$

From (1.7), we have

$$\begin{aligned}\frac{d}{dr}[ru'(ru' + 2)] &= 2(ru')'\,(ru' + 1)\\ &= -2rK(r)\, e^{2u(r)}[ru' + 1].\end{aligned}$$

Hence, we have

$$\begin{aligned}P(r; u(r)) &= ru'(r)[ru' + 2] + r^2 K(r)\, e^{2u(r)}\\
&= r^2 K(r)\, e^{2u(r)} - 2\int_0^r sK(s)\, e^{2u(s)}\, ds - 2\int_0^r s^2 K(s)\, e^{2u(s)} u'(s)\, ds\\
&= \left[G(r) + 2\int_0^r sK(s)\, ds\right] e^{2u(r)} - 2\int_0^r sK(s)\, e^{2u(s)}\, ds\\
&\quad - 2\int_0^r \left[G(s) + 2\int_0^s tK(t)\, dt\right] e^{2u(s)} u'(s)\, ds\\
&= G(r)\, e^{2u(r)} + 2e^{2u(r)} \int_0^r sK(s)\, ds - 2\int_0^r G(s)\, e^{2u(s)} u'(s)\, ds\\
&\quad - 2\int_0^r sK(s)\, e^{2u(s)}\, ds - 4\int_0^r \left(\int_0^s tK(t)\, dt\right) e^{2u(s)} u'(s)\, ds\\
&= G(r)\, e^{2u(r)} - 2\int_0^r G(s)\, e^{2u(s)} u'(s)\, ds.\end{aligned}$$

Since $G(r) \geqslant 0$, $G(r) \not\equiv 0$ and $u' < 0$, we conclude that $P(r; u(r))$ is non-negative and $P(r; u(r)) > 0$ for r large. From (2.50), there exists a sequence $\{r_i\}$, such that $r_i \to \infty$ as $r \to \infty$ and $r_i^2 K(r_i)\, e^{2u(r_i)} \to 0$ as $i \to \infty$. Thus from (2.50) and (2.51) we conclude that

$$\alpha(a)[\alpha(a)+2] = -2 \int_0^\infty G(s)\, e^{2u(s)} u'(s)\, ds > 0 \tag{2.52}$$

with $\alpha(a) < 0$. Hence $\alpha(a) < -2$. This proves Theorem 1.6. Q.E.D.

Proof of Theorem 1.7. Using the identity (2.51) and the fact that $G(r) \leqslant 0$ and $G(r) \not\equiv 0$, we conclude that $P(r; u(r)) < 0$ for r large. Then we conclude that

$$\alpha(a)[\alpha(a)+2] < 0 \tag{2.53}$$

with $\alpha(a) < 0$. But since $K(r)$ is bounded by two positive constants and $\int_0^\infty sK(s)\, e^{2u(s)}\, ds < \infty$, we conclude that $-2 < \alpha(a) < -1$. This proves Theorem 1.7. Q.E.D.

Proof of Theorem 1.8. We divide the proof into several steps.

Step 1. There exists a constant M so large that $\alpha(a) > -2$ for all $a \leqslant -M$.

To prove this claim, we first let $v(r, a) = u(r, a) - a$. Then $v(r, a)$ satisfies

$$\begin{cases} v'' + \dfrac{1}{r} v' + K(r)\, e^{2a} e^{2v} = 0, \\ v(0) = 0,\ v'(0) = 0. \end{cases}$$

Now since $v(r, a) \leqslant 0$ and $K(r)\, e^{2a} e^{2v} \to 0$ uniformly on any compact interval of $[0, \infty)$, we have $v(r, a) \to 0$ and $v'(r, a) \to 0$ uniformly on any $[0, r]$ as $a \to -\infty$. From (2.51), we have

$$e^{-2a} P(r; u(r, a)) = G(r)\, e^{2v(r, a)} - 2 \int_0^r G(s)\, e^{2v(s, a)} v'(s, a)\, ds \to G(r)$$

uniformly on any $[0, r]$ as $a \to -\infty$.

Thus there exists $M > 0$ such that

$$P(r; u(r, a))\,|_{r=R} < 0$$

for all $a \leqslant -M$. For $r \geqslant R$, since $G(r) \leqslant G(R)$, we have

$$\begin{aligned}
&P(r; u(r, a)) - P(R; u(R, a)) \\
&\qquad = [G(r) - G(R)]\, e^{2u(r, a)} - 2 \int_R^r [G(s) - G(R)]\, e^{2u(s, a)} u'(s, a)\, ds \\
&\qquad \leqslant 0.
\end{aligned}$$

Hence $P(r; u(r, a)) \leqslant P(R; u(R, a)) < 0$ for all $r \geqslant R$ and $a \leqslant -M$. This proves that $\alpha(a) > -2$ for $a \leqslant -M$.

Step 2. There exists a constant N so large that $\alpha(a) < -2$ for all $a \geqslant N$.

To prove this, we use the results of Cheng and Smoller [CS, Theorems 2.6 and 2.10] to conclude that $\delta u'(\delta, a) = -2 - CK_1(0)\, e^{-ka} +$ Higher order terms for some $\delta > 0$, $C > 0$ and $a \geqslant N$, where N is sufficiently large. Since $(ru'(r, a))' = -rK(r)\, e^{2u(r, a)} < 0$ for $r > 0$, we conclude that $\alpha(a) < -2$ for $a \geqslant N$.

Step 3. From Steps 1 and 2 and the continuity of $\alpha(a)$, we conclude that there is an a_0 such that $\alpha(a_0) = -2$. Q.E.D.

Proof of Theorem 1.9. As in the proof of Theorem 1.8, first we prove that there exists an $M > 0$ such that $\alpha(a) < -2$ for all $a \leqslant -M$. We omit this part of the proof. Next, we use the result of Cheng and Smoller [CS, Theorem 2.6] to conclude that

$$\begin{cases} \delta u'(\delta, a) = -2 + CK_1(0)\, e^{-ka} + O(e^{-2a} a^2), \\ u(\delta, a) = -a + O(1), \end{cases}$$

for a sufficiently large and some $\delta > 0$, where $C > 0$ is a constant independent of a and δ. For $r \geqslant \delta$, we have

$$ru'(r) = \delta u'(\delta) - \int_\delta^r sK(s)\, e^{2u(s)}\, ds,$$

$$u(r) = u(\delta) + \delta u'(\delta) \log\left(\frac{r}{\delta}\right) - \int_\delta^r s \log\left(\frac{r}{s}\right) K(s)\, e^{2u(s)}\, ds.$$

Hence we obtain

$$u(r) = -a - (2 - \varepsilon) \log r + O(1) \qquad \text{for} \quad r \geqslant \delta,$$

$$\lim_{r \to \infty} ru'(r) = -2 + CK_1(0)\, e^{-ka} + O(e^{-2a} a^2).$$

This proves that there exists an $N > 0$ such that $\alpha(a) > -2$ for $a \geqslant N$.

Combining these results, we conclude that there exists an a_0 such that $\alpha(a_0) = -2$. The proof is complete. Q.E.D.

3. EXAMPLES

In this section, we will construct explicitly all radial solutions for Theorems 1.2 and 1.4.

Proof of Theorem 1.2. For $-1 < p$, we let

$$u_\alpha(r) = \tfrac{1}{2}\log[4(1+p)^2 B] - \log[1 + Br^{2(1+p)}], \qquad r \in [0, 1] \tag{3.1}$$

$$\begin{aligned} u_\alpha(r) = {} & \tfrac{1}{2}\log[4A_1^2 B_1] + (A_1 - 1 - q)\log r \\ & - \log[1 + B_1 r^{2A_1}], \qquad r \in [1, \infty), \end{aligned} \tag{3.2}$$

where $B > 0$ is a constant and $\alpha = -A_1 - 1 - q$. It is not difficult to verify that u_α is a C^2 solution of (1.5) with $u_\alpha(0) = \frac{1}{2}\log[4(1+p)^2 B]$ provided that

$$A_1^2 = \frac{[B(q-1-2p) + (q+1)]^2 + 4(1+p)^2 B}{(1+B)^2}, \qquad A_1 > 0, \tag{3.3}$$

and

$$B_2 = \frac{A_1(1+B) - [B(q-1-2p) + (q+1)]}{A_1(1+B) + [B(q-1-2p) + (q+1)]}. \tag{3.4}$$

Since $u_\alpha(0) = \frac{1}{2}\log[4(1+p)^2 B]$, we see that $B > 0$ exhausts all radial solutions. It is easy to see that u_α satisfies (1.8) with $\alpha(B) = -A_1(B) - 1 - q$. Now we can easily verify that

$$\frac{dA_1^2(B)}{dB} = \frac{4(1+p)(p-q)}{(1+B)^2}. \tag{3.5}$$

Hence $A_1(B)$ is an increasing function of B if $q < p$, a constant function if $q = p$, and a decreasing function if $q > p$. Hence $\alpha(B)$ is also a monotonic function of B.

From (3.3), it is easy to see that

$$\lim_{B \to 0} A_1(B) = |q+1| \qquad \text{and} \qquad \lim_{B \to \infty} A_1(B) = |q-1-2p|. \tag{3.6}$$

Thus we have

$$\begin{aligned} &-2-2p<\alpha(B)<\min\{0,\ -2-2q\} && \text{if} \quad q<p \\ &\alpha(B)=-2(1+p) && \text{if} \quad q=p \\ &-2-2q<\alpha(B)<\min\{-2(1+p),\ -2(q-p)\} && \text{if} \quad q>p. \end{aligned}$$

This proves Theorem 1.2. Q.E.D.

Proof of Theorem 1.4. We let

$$u(r, B)=\frac{1}{2}\log\left(\frac{4B}{K_0}\right)-\log[1+Br^2], \qquad \text{for} \quad r\in[0,1], \tag{3.7}$$

$$\begin{aligned} u(r, B)=&\frac{1}{2}\log(4A_1^2B_1)-(A_1+1-m)\log r \\ &-\log[1-B_1r^{-2A_1}], \qquad \text{for} \quad r\in[1,\infty), \end{aligned} \tag{3.8}$$

where $B>0$ is a constant and A_1, B_1 are determined by the requirement that $u(r, B)$ is C^1 at $r=1$. The conditions that $u(r, B)$ is C^1 at $r=1$ are

$$\frac{1}{2}\log[4A_1^2B_1]-\log[1-B_1]=\frac{1}{2}\log\left(\frac{4B}{K_0}\right)-\log(1+B), \tag{3.9}$$

$$-(A_1+1-m)-\frac{2A_1B_1}{1-B_1}=-\frac{2B}{1+B}. \tag{3.10}$$

By a straightforward computation, $u(r, B)$ satisfies (1.7) if both (3.9) and (3.10) hold. It is easy to see from (3.8) that $A_1>0$ and $0<B_1<1$ are necessary conditions to make $u(r, B)$ exist on $[0,\infty)$. From (3.9) and (3.10) we obtain

$$\frac{A_1^2B_1}{(1-B_1)^2}=\frac{B}{K_0(1+B)^2}, \tag{3.11}$$

$$\frac{A_1(1+B_1)}{1-B_1}=\frac{B(1+m)-(1-m)}{1+B}. \tag{3.12}$$

Thus we conclude that

$$B(1+m)-(1-m)>0. \tag{3.13}$$

Using (3.11) and (3.12) we obtain

$$\frac{(1+B_1)^2}{B_1} = \frac{K_0[B(1+m)-(1-m)]^2}{B}, \tag{3.14}$$

$$A_1^2 = \frac{K_0[B(1+m)-(1-m)]^2 - 4B}{K_0(1+B)^2}, \tag{3.15}$$

$$B_1 = \frac{[B(1+m)-(1-m)] - A_1(1+B)}{[B(1+m)-(1-m)] + A_1(1+B)}. \tag{3.16}$$

Now we let

$$B_{\pm} = \frac{[K_0(1-m^2)+2] \pm 2\sqrt{K_0(1-m^2)+1}}{K(1+m)^2} \qquad \text{if} \quad -1 < m < \sqrt{\frac{K_0+1}{K_0}}, \tag{3.17}$$

$$\tilde{B} = \frac{K_0(1-m)+1}{K_0(1+m)+1} \qquad \text{if} \quad m \geqslant \sqrt{\frac{K_0+1}{K_0}}. \tag{3.18}$$

We divide the discussion into several cases.

Case 1. $m \leqslant -1$.

In this case, $B(1+m)-(1-m)<0$ for all $B>0$. From (3.13) and (3.12), we conclude that $B_1>1$ and that $u(r, B)$ blows up at finite r for every $B>0$. Thus there are no entrie radial solutions. This proves (i).

Case 2. $-1<m\leqslant 1$.

In this case, it is easy to verify that $0 \leqslant B_- \leqslant (1-m)/(1+m) < B_+$. Thus from (3.13) and (3.15), we conclude that if $0<B<B_+$, then either $B_1>1$ or $A_1^2<0$. Hence $u(r, B)$ also blows up at finite r. For each B, $B_+ \leqslant B < \infty$, $u(r, B)$ exists on $[0, \infty)$. It is easy to check that $u(r, B) = \alpha(B) \log r + C + o(1)$ as $r \to \infty$ for $B_+ < B$, where $\alpha(B) = -[A_1(B)+1-m]$. From (3.15) we conclude that $\alpha(B)$ is a monotonically decreasing function of B satisfying

$$\lim_{B \to B_+} \alpha(B) = m-1 \qquad \text{and} \qquad \lim_{B \to \infty} \alpha(B) = -2. \tag{3.19}$$

It is also interesting to know that for $B = B_+$, we have

$$u(r, B_+) = \frac{1}{2} \log\left(\frac{4B_+}{K_0}\right) - \log[1 + B_+ r^2], \qquad \text{for} \quad r \in [0, 1]$$

$$u(r, B_+) = \frac{1}{2} \log\left(\frac{4B_+}{K_0(1 + B_+)^2}\right) + (m-1) \log r$$
$$- \log[1 + B_1 \log r], \qquad \text{for} \quad r \in [1, \infty),$$

where

$$B_1^2 = \frac{4B_+}{K_0(1 + B_+)^2}.$$

Thus we have

$$u(r, B_+) = \lim_{B \to B_+} u(r, B) = (m-1) \log r - \log(\log r) + O(1)$$

as $r \to \infty$. The range of α, $-2 < \alpha < m - 1 = \alpha_1(K)$, and the result $u(r, B_+) = \lim_{B \to B_+} u(r, B)$ are exactly the result obtained in Cheng and Lin [CLn2]. This proves (ii).

Case 3. $1 < m < \sqrt{(K_0 + 1)/K_0}$.

In this case, (3.13) always holds. Thus $A_1^2(B) > 0$ for $0 < B < B_-$ and $B_+ < B$ and $A_1^2(B) < 0$ for $B_- < B < B_+$. We conclude that

(i) If $0 < B \leqslant B_-$, then $u(r, B)$ exists on $[0, \infty)$ and satisfies $u(r, B) = \alpha(B) \log r + C + o(1)$ as $r \to \infty$, where $\alpha(B) = -(A_1 + 1 - m)$ is monotonically increasing in $0 < B < B_+$ satisfying $\lim_{B \to 0} \alpha(B) = 0$ and $\lim_{B \to B_-} \alpha(B) = m - 1$. As in the previous case, we have

$$u(r, B_-) = (m-1) \log r - \log(\log r) + O(1)$$

as $r \to \infty$. It is also easy to see that $u(r; B) < u(r; B')$ for all $0 < B < B' \leqslant B_-$.

(ii) If $B_- < B < B_+$, then $A_1^2 < 0$, we conclude that $u(r, B)$ blows up at finite r.

(iii) If $B_+ \leqslant B < \infty$, then $u(r, B)$ exists on $[0, \infty)$ and satisfies $u(r, B) = \alpha(B) \log r + C + o(1)$ as $r \to \infty$, where $\alpha(B)$ is monotonically decreasing in $B_+ < B < \infty$ satisfying $\lim_{B \to B_+} \alpha(B) = m - 1$ and $\lim_{B \to \infty} \alpha(B) = -2$. Also

$$u(r, B_+) = (m-1) \log r - \log(\log r) + O(1)$$

as $r \to \infty$.

It is also easy to verify that if $\alpha(B')=\alpha(B'')$ for $0<B'<B_-$ and $B_+<B''<\infty$, then $u(r, B')<u(r, B'')$ for all r. This proves (iii).

Case 4. $m=\sqrt{(K_0+1)/K_0}$.

This case is similar to that of Case 3 with $B_+=B_-=\tilde{B}$, where $\tilde{B}$ is defined in (3.18). Thus, $u(r, B)$ exists for all $0<B$, $\alpha(B)$ is monotonically increasing in $0<B<\tilde{B}$, and is monotonically decreasing in $\tilde{B}<B<\infty$ satisfying $\lim_{B\to 0}\alpha(B)=0$, $\lim_{B\to\tilde{B}}\alpha(B)=m-1$, and $\lim_{B\to\infty}\alpha(B)=-2$. Precisely, we have

$$u(r, \tilde{B})=(m-1)\log r-\log(\log r)+O(1)$$

as $r\to\infty$. This proves (iv).

Case 5. $\sqrt{(K_0+1)/K_0}<m<(K_0+1)/K_0$.

In this case, $\int_{\mathbf{R}^2}K(r)\,dx<0$. We conclude that $u(r, B)$ exists on $[0, \infty)$ for all $B>0$ and $u(r, B)=\alpha\log r+O(1)$ as $r\to\infty$. Here $\alpha(B)$ is monotonically increasing in $0<B<\tilde{B}$ and is monotonically decreasing in $\tilde{B}<B<\infty$, $\alpha(B)\to 0$ as $B\to 0$, and $\alpha(B)\to -2$ as $B\to\infty$. It is interesting to note that $0<\alpha(\tilde{B})<m-1$. This proves (v).

Case 6. $(K+1)/K\leqslant m$.

In this case, $\int_{\mathbf{R}^2}K(r)\,dx\geqslant 0$. Thus, $u(r, B)$ exists for all $0<B<\infty$ and $u(r, B)=\alpha(B)\log r+C+o(1)$ as $r\to\infty$ for all $B>0$, where $\alpha(B)$ is monotonically decreasing in $0<B<\infty$ satisfying $\alpha(B)\to 0$ as $B\to 0$ and $\alpha(B)\to -2$ as $B\to\infty$. This proves (vi). ∎

4. PROOFS OF THEOREM 1.10

In this section, we construct explicitly $K_1(r)$ and $K_2(r)$ and the corresponding solutions to find out $\alpha_1(a)$ and $\alpha_2(a)$. We let $\eta>0$ be a fixed number and γ is a large fixed number. Let $n\geqslant 1$ be a fixed positive integer. We let $r_1>0$ and $K_1>0$ be fixed and

$$r_{i+1}=\gamma^i r_1, \qquad i=1, 2, ..., (n-1), \tag{4.1}$$

$$K_{i+1}=\eta^i K_1, \qquad i=1, 2, ..., n. \tag{4.2}$$

We define $K_1(r)=K_i$ for $r\in[r_{i-1}, r_i]$, $i=1, 2, ..., n+1$ for some $\eta>1$ and similarly for $K_2(r)$ with $\eta<1$.

Now for $r \in [r_{i-1}, r_i]$, we let

$$u(r) = \frac{1}{2} \log \frac{4A_i^2 B_i}{K_i} + (A_i - 1) \log r - \log[1 + B_i r^{2A_i}] \tag{4.3}$$

for $i = 1, 2, ..., n+1$, where we let $r_0 = 0$ and $r_{n+1} = \infty$. Then it is easy to see that $u(r)$ is a C^1 solution of (1.7) with $u(0) = a = \frac{1}{2} \log(4B_1/K_1)$ and $u(r) = \alpha(a) \log r + O(1)$ at $r = \infty$ where $\alpha(a) = -(A_{n+1} + 1)$, provided that $A_1 = 1$, $B_1 > 0$, $K(r) = K_i$ for $r \in [r_{i-1}, r_i]$, $i = 1, 2, ..., n+1$ and

$$A_{i+1} = A_i G(B_i r_i^{2A_i}), \qquad i = 1, 2, ..., n, \tag{4.4}$$

$$B_{i+1} r_i^{2A_{i+1}} = H(B_i r_i^{2A_i}), \qquad i = 1, 2, ..., n, \tag{4.5}$$

where

$$G(z) = \left\{1 + \frac{4(\eta - 1) z}{(1+z)^2}\right\}^{1/2}, \tag{4.6}$$

$$H(z) = \frac{G(z)(1+z) - (1-z)}{G(z)(1+z) + (1-z)}. \tag{4.7}$$

From the definitions, it is easy to see that $K_1(r)$ is nondecreasing and $K_2(r)$ is nonincreasing. For $\eta > 1$, $G(z)$ is increasing for $0 < z < 1$ and decreasing for $1 < z < \infty$ with a maximum at $z = 1$. It is easy to see that $G(0) = 1$, $G(1) = \eta^{1/2}$, and $G(\infty) = 1$. It is also easy to see that $0 < H(z) < \infty$ and $H(0) = 0$, $H(\infty) = \infty$. Thus if $\eta > 1$ and γ is very large, then

$$\begin{aligned} \alpha(a) &= -(A_{n+1} + 1) \\ &= -1 - G(B_1 r_1^2)\, G(B_2 r_2^{2A_2}) \cdots G(B_n r_n^{2A_n}) \end{aligned} \tag{4.8}$$

has local minima near the n points of B for $B_i r_i^{2A_i} = 1$, $i = 1, 2, ..., n$ and the local minima of $\alpha(a)$ are near $-1 - \eta^{1/2} < -2$.

Similarly, if $\eta < 1$ and γ is very large, than $\alpha(a)$ has local maxima near the n points mentioned above and the local maxima of $\alpha(a)$ are near $-1 - \eta^{1/2}$, $-2 < -1 - \eta^{1/2} < -1$.

We regard the proof of Theorem 1.10 as complete. Q.E.D.

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